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On pseudo 2-factors

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ABSTRACT

We show that a graph with minimum degree δ, independence number $\alpha \geq \delta$ and without isolated vertices, possesses a partition by vertex-disjoint cycles and at most $\alpha - \delta + 1$ edges or vertices.

1. Introduction

Throughout this paper, we consider only finite simple graphs $G = (V, E)$. We denote by δ the minimum degree of the considered graph and by α its independence number. Let C be a cycle with a prescribed orientation. Let u and v be two vertices on the cycle C, we denote by $]u, v[_C$ the segment of C, following the orientation and delimited by u and v, u and v excluded. If it does not matter whether u and v are included or not then we replace the braces by brackets. We denote by $d_C(u, v)$ the distance between u and v on the cycle C. The join of two disjoint graphs G_1 and G_2 is denoted by $G_1 + G_2$ and is the graph obtained by joining each vertex of G_1 to each vertex of G_2. For a positive integer p, the graph pG consists of p vertex-disjoint copies of G. For concepts not defined here we refer to [2].

A covering of a graph G is a family of elementary cycles of G such that each vertex of G lies in at least one cycle of this family. In the literature there are many results dealing with coverings of graphs, particularly by disjoint cycles. A summary of results on independent cycles can be found in [5,7]. In particular, there are some results, involving degree conditions for the existence of k disjoint cycles and s edges, where k and s are fixed [1] or k disjoint cycles and a prescribed forest of size s [9,4].

We define a *pseudo 2-factor* of G as a partition of V by a family of vertex disjoint cycles, edges or vertices. The cardinality of this family will be called the size of the pseudo 2-factor.

These two notions as different as they appear generalize in some sense the same concept, namely that of 2-factors. Recall that a 2-factor of G is a 2-regular spanning subgraph of G. Clearly, if the cycles taken in a covering of G are vertex-disjoint then this covering is a 2-factor, and, if a pseudo 2-factor of G contains only cycles then it is a 2-factor. This case occurs when the independence number of G is at most $\delta - 1$ (see [8]). In [8], Niessen has also showed that graphs with independence number $\alpha = \delta$ containing no 2-factor are the graphs $H + \delta K_2$, where H is a graph of order $\delta - 1$. We check easily that such graphs possess a pseudo 2-factor (of size at most α) in which all the components are cycles but one.

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Our work was inspired by Kouider's paper [6] and motivated by the desire to answer the following question: What is the number of components which are edges or vertices in a pseudo 2-factor of a graph with $\alpha > \delta$?

We investigate relation between the minimum degree, the independence number and the number of edges or vertices in a pseudo 2-factor. The main result of this paper is the following, answering thereby the set question, and including the case $\alpha = \delta$ too:

Theorem 1. *Let G be a graph without isolated vertices, with minimum degree δ and independence number $\alpha \geq \delta$, then there exists a pseudo 2-factor of G with at most $\alpha - \delta + 1$ components that are edges or vertices.*

The bound given in the theorem above is best possible. To see that, consider the graph $G = H + pK_2$ where $p \geq |H| + 1$ (whatever the graph H is). This graph has minimum degree $\delta = |H| + 1 \geq 2$, independence number $\alpha = p$ and possesses a pseudo 2-factor with exactly $\alpha - \delta + 1$ edges and without isolated vertices. It is easy to check that no pseudo 2-factor with less edges or vertices can be found for such a graph. There also exists graphs with $\delta = 1$ for which the bound is reached. As an example, take a graph H of order n and a independent set of order n. Attach exactly a vertex of H to exactly a vertex of this independent set. The graph obtained has independence number $\alpha = n$ and possesses no pseudo 2-factor with less than n edges or vertices.

Niessen's result for graphs with $\alpha = \delta$ derives naturally from the theorem above.

Corollary 1 (*[8]*). *Let G be a graph with independence number α and minimum degree δ such that $\alpha = \delta$. Then, G possesses a pseudo 2-factor containing at most one component which is an edge or a vertex.*

In addition, Theorem 1 gives a lower bound for the number of vertices that are covered by vertex disjoint cycles.

Corollary 2. *Let G be a graph with independence number α and minimum degree $\delta \leq \alpha$. then at least* $\max(2\delta - 2, n + 2\delta - 2(\alpha + 1))$ *vertices of G can be covered by vertex disjoint cycles.*

The bound given above is reached for the graphs of type $H + pK_2$ with $p \geq |H| + 1$ defined above.

2. Pseudo-factors, minimum degree and independence number

We begin by the simplest case which is when the graph G has minimum degree δ at most 1. In this case, the theorem above is a consequence of the following proposition which has already been established, particularly by Bondy [3].

Proposition 1. *Let G be a graph with independence number α, then G possesses a pseudo 2-factor of size at most α.*

Proof. The proof is done by induction on α.

For $\alpha = 1$, it is true.

Suppose that $\alpha \geq 2$, let P be a longest path in G and let x be an end-vertex of P.

(1) If x has degree 1, then if we remove x and its neighbor x' we get $\alpha(G - \{x, x'\}) \leq \alpha - 1$. By induction hypothesis, $G - \{x, x'\}$ possesses a pseudo 2-factor containing at most $\alpha - 1$ cycles edges or vertices and adding $\{x, x'\}$ (and the edge joining them) we obtain a pseudo 2-factor of G of size at most α.

(2) If x has degree at least 2, then consider y the farthest neighbor of x on P and let C be the cycle formed by the segment $[x, y]_P$ and the edge $e = (x, y)$. We have that $\alpha(G - C) \leq \alpha - 1$. By induction hypothesis, $G - C$ has a pseudo 2-factor of size at most $\alpha - 1$ and it follows that there exists a pseudo 2-factor of G containing at most α cycles, edges or vertices. ∎

From now, let G be a graph with minimum degree $\delta \geq 2$ and independence number $\alpha \geq \delta$. Let $\mathcal{F}$ be a family $C_1, \ldots, C_r$ of vertex disjoint cycles of G. Denote by F the smallest component of $G - \bigcup_{i=1}^r C_i$, set $W = G - (F \cup (\bigcup_{i=1}^r C_i))$ and choose a family $\mathcal{F}$ of cycles for which:

(a) $\alpha(G - \bigcup_{i=1}^r C_i)$ as small as possible;
(b) subject to (a), r as small as possible;
(c) subject to (a) and (b), F as small as possible.

Notice that a family of cycles satisfying the conditions above exists. Indeed, since $\delta \geq 2$, then there exists at least a cycle C such that $\alpha(G - C) < \alpha$. The cycle C can be obtained using the construction with longest paths described in the proof of Proposition 1.

Furthermore each component of $W \cup F$ has minimum degree at most 1. Indeed if a component A of $W \cup F$, has minimum degree δ_A at least 2, then, a longest path P in A provides a cycle C which verifies $\alpha(A - C) < \alpha(A)$ and $\alpha(G - \bigcup_{i=1}^r C_i) > \alpha(G - [\bigcup_{i=1}^r C_i \cup C])$. This contradicts (a) in the definition of $\mathcal{F}$.

We also remark that under conditions (a) and (b), each cycle of the family $\mathcal{F}$ verifies: $\alpha(W \cup F \cup C_i) > \alpha(W \cup F)$, for $i = 1, \ldots, r$. Indeed, if for some k $(1 \leq k \leq r)$, we have $\alpha(W \cup F \cup C_k) = \alpha(W \cup F)$, then the family $\mathcal{F}'$ of cycles $\{C_i\}_{i \neq k}$, would verify condition (a) and would contain less cycles than $\mathcal{F}$, contradicting condition (b) and thus the choice of $\mathcal{F}$.

Moreover, we shall show that if all the cycles of $\mathcal{F}$ are added to $W \cup F$ then the independence number of this latter will increase by at least $\delta - 1$. More precisely, we show the following result:

Theorem 2. *Let G be a graph with minimum degree $\delta \geq 2$ and independence number $\alpha \geq \delta$. Then there exists a pseudo 2-factor of G such that $C_1, \ldots, C_r$ are the cycles of this pseudo 2-factor with*

$$\alpha\left(G - \bigcup_{i=1}^{r} C_i\right) \leq \alpha - (\delta - 1).$$

This implies Theorem 1.

Proof. We need some further notations. Denote by $C_1, \ldots, C_{r_1}$ the cycles of $\mathcal{F}$ on which F possesses at least two neighbors, by $C_{r_1+1}, \ldots, C_{r_2}$ those on which F possesses exactly one neighbor and by $C_{r_2+1}, \ldots, C_r$ those on which F has no neighbor. Denote by c_i the neighbor of F on a cycle C_i, for $r_1 + 1 \leq i \leq r_2$ and with respect to a specific orientation of C_i, for each i, $1 \leq i \leq r_1$, denote by $c_i^1, \ldots, c_i^{m_i}$ the neighbors of F, in this order, on C_i. ■

Lemma 1. *Let k and l be two integers with $1 \leq k \leq l \leq r_2$. Let C' be a cycle which contains the neighbors of F on $C_l \cup C_k$, at least a vertex of F, and such that $V(C') \subset V((C_l \cup C_k) \cup F \cup W)$. Set $W' = G - (\bigcup_{i \neq l, i \neq k} C_i \cup F \cup C')$. Then $\alpha(W') > \alpha(W)$.*

Proof of Lemma 1. Set $F_0 = F - C'$ and let $\mathcal{F}'$ be the family of cycles $\{C_{i, i \neq l, i \neq k}, C'\}$.

(1) If $k \neq l$, then $\mathcal{F}'$ contains less cycles than $\mathcal{F}$ and hence must not verify condition (a), so:

$\alpha(G - (\bigcup_{i \neq l, i \neq k} C_i \cup C')) > \alpha(G - \bigcup_{i=1}^{r} C_i) = \alpha(W) + \alpha(F)$. On another hand, $\alpha(G - (\bigcup_{i \neq l, i \neq k} C_i \cup C')) = \alpha(W') + \alpha(F_0)$, because F has no neighbor in $(C_k \cup C_l) - C'$. It follows that $\alpha(W') > \alpha(W)$ (because $\alpha(F) \geq \alpha(F_0)$).

(2) If $k = l$, then $\mathcal{F}'$ and $\mathcal{F}$ have the same number of cycles. Two cases may occur.

(i) If $F_0 = \emptyset$, then by condition (a) on $\mathcal{F}$, we have

$\alpha(W') = \alpha(G - (\bigcup_{i \neq l, i \neq k} C_i \cup C')) \geq \alpha(G - \bigcup_{i=1}^{r} C_i) = \alpha(W) + \alpha(F)$ so $\alpha(W') \geq \alpha(W) + 1$.

(ii) If $F_0 \neq \emptyset$, then F_0 is smaller than F. The family of cycles $\mathcal{F}'$ verifies (b) and consequently do not verify condition (a), otherwise we get a contradiction with condition (c), hence:

$\alpha(W') + \alpha(F_0) = \alpha(G - (\bigcup_{i \neq l, i \neq k} C_i \cup C')) > \alpha(G - \bigcup_{i=1}^{r} C_i) = \alpha(W) + \alpha(F)$. As $\alpha(F_0) \leq \alpha(F)$ thus $\alpha(W') > \alpha(W)$. □

Let V be an interval on a cycle C_k, for $1 \leq k \leq r_2$.

We say that the interval V has *property Θ* if and only if $\alpha(W \cup F \cup V) = \alpha(W \cup F)$. We say that two different intervals V and V' are *path-independent* if there exists no path internally disjoint from $\cup_{i=1}^{r} C_i \cup F$ joining a vertex of V to a vertex of V'. We say that t intervals are path-independent if they are pairwise path-independent. The following lemma will be intensively used:

Lemma 2. *Let V and V' be two different intervals not neighbors of F ($V \cup V' \subset C_k \cup C_l$, $1 \leq k \leq l \leq r_2$). Suppose that both V and V' have property Θ, then*

(1) *If V and V' are path-independent then $V \cup V'$ has property Θ.*
(2) *V and V' are path-independent.*
(3) *More generally if t disjoint intervals $V^{(1)}, V^{(2)}, \ldots., V^{(t)}$ ($t \geq 2$) have property Θ then $V^{(1)} \cup V^{(2)} \cup \cdots \cup V^{(t)}$ has property Θ*

Proof of Lemma 2. (1) Let H (respectively H') be the union of the components of W that have a neighbor on V (respectively on V'). By hypothesis, as V and V' are path independent, $H \cap H' = \emptyset$, hence $W - (H \cup H')$, $H \cup V$, $H' \cup V'$ form a partition of $W \cup V \cup V'$ and it follows that $\alpha(W \cup V \cup V') = \alpha(W - (H \cup H')) + \alpha(H \cup V) + \alpha(H' \cup V')$. Furthermore, because of property Θ, $\alpha(H \cup V) = \alpha(H)$ and $\alpha(H' \cup V') = \alpha(H')$. So $\alpha(W \cup V \cup V') = \alpha(W - (H \cup H')) + \alpha(H) + \alpha(H') = \alpha(W)$.

(2) First, denote by $P^i_{jj'}$ a path with internal vertices in F joining the vertices c_i^j and $c_i^{j'}$ belonging to a same cycle C_i ($1 \leq i \leq r_1$), or simply $P_{jj'}$ if the vertices joined belong to different cycles ($1 \leq i \leq r_1$). Suppose that there exists a path internally disjoint from $\cup_{i=1}^{r} C_i \cup F$ joining V and V'. It implies that either there is a path with internal vertices in W joining a vertex in V to a vertex in V' or that a vertex in V is adjacent to a vertex in V'. We distinguish two cases according to the fact that V and V' are on the same cycle or not.

(i) Suppose that V and V' belong to a same cycle C_k, $1 \leq k \leq r_1$. Put $V =]c_k^j, v_k)_{C_k}$ and $V' =]c_k^{j'}, v'_k)_{C_k}$ (with $1 \leq j < j' \leq m_k$). Let $x \in V$ and $x' \in V'$ be two vertices joined by a path of $W \cup \{x\} \cup \{x'\}$ and chosen so as to minimize the sum of lengths $d_{C_k}(c_k^j, x)$ and $d_{C_k}(c_k^{j'}, x')$. Note that by this choice the segments $]c_k^j, x[_{C_k}$ and $]c_k^{j'}, x'[_{C_k}$ are path-independent, furthermore they do both have property Θ (as they are, respectively, included in V and V'). So by (1), we have $\alpha(W \cup]c_k^j, x[_{C_k} \cup]c_k^{j'}, x'[_{C_k}) = \alpha(W) (\star)$

- If x and x' are adjacent, then taking $C' = c_k^j P^k_{j,j'} \overleftarrow{[c_k^{j'}, x]}_{C_k} (x, x')[x', c_k^j]_{C_k}$ in Lemma 1, we obtain $\alpha(W) < \alpha(W \cup]c_k^j, x[_{C_k} \cup]c_k^{j'}, x'[_{C_k})$ and hence a contradiction with $(\star)$.

- If there exists a path Q with internal vertices in W joining x and x', then taking $C' = c_k^j P_{j,j'}^k \overleftarrow{[c_k^{j'}, x]}_{C_k} Q[x', c_k^j]_{C_k}$ in Lemma 1, we obtain $\alpha(W) < \alpha(W_0 \cup]c_k^j, x[_{C_k} \cup]c_k^{j'}, x'[_{C_k}) \le \alpha(W \cup]c_k^j, x[_{C_k} \cup]c_k^{j'}, x'[_{C_k})$, where $W_0 = W - Q$. So again, a contradiction with $(\star)$

(ii) Suppose that V and V' are on different cycles C_k and C_l ($1 \le k < l \le r_2$). Same as (a), using Lemma 1 and the first part of Lemma 2.

(3) Let $V^{(1)}, V^{(2)}, \dots V^{(t)}$ be t ($t \ge 2$) different intervals having property Θ.

By (2) of Lemma 2, they are path-independent. By induction on t we show that property Θ is conserved in $V^{(1)} \cup V^{(2)} \cup \dots \cup V^{(t)}$. We set $V_1 = V^{(1)} \cup V^{(2)} \cup \dots \cup V^{(t-1)}$, $V_2 = V^{(t)}$ and the result follows by the proof of Lemma 2(1). □

We already know that by conditions on the chosen family of cycles, the addition of a cycle C_i of $\mathcal{F}$ to $W \cup F$ increases the independence number of $W \cup F$ by at least 1. We show now that this augmentation can be more significant if F possesses more than a neighbor on the added cycle, in other words if $1 \le i \le r_1$. To see that, it suffices to consider the segments of C_i ($1 \le i \le r_1$) included between two consecutive neighbors of F. They are of type $]c_i^j, c_i^{j+1}[_{C_i}$ where j is taken module m_i ($1 \le j \le m_i$). We claim that these segments do not have property Θ. Let $P_{jj'}^i$ and $P_{jj'}$ be as defined in the proof of Lemma 2.

Lemma 3. *For all i, j, $1 \le i \le r_1$, and, $1 \le j \le m_i$ where j is taken modulo m_i, we have*

(1) $]c_i^j, c_i^{j+1}[_{C_i} \neq \emptyset$.

(2) $\alpha(W \cup]c_i^j, c_i^{j+1}[_{C_i}) > \alpha(W)$.

Proof of Lemma 3. (1) Suppose to the contrary that a segment $]c_l^j, c_l^{j+1}[_{C_l} = \emptyset$ on some cycle C_l. Then, $C' = c_l^j P_{j,j+1}^l [c_l^{j+1}, c_l^j]_{C_l}$ gives a contradiction with the definition of the family $\mathcal{F}$.

(2) Setting $C' = c_i^j P_{j,j+1}^i [c_i^{j+1}, c_i^j]_{C_i}$, in Lemma 1, we obtain $\alpha(W') = \alpha(W \cup]c_i^j, c_i^{j+1}[_{C_i}) > \alpha(W)$. □

Let u_i^j be the first vertex in $]c_i^j, c_i^{j+1}[_{C_i}$ such that $\alpha(W \cup]c_i^j, u_i^j]_{C_i}) > \alpha(W)$.

Lemma 4. *There is no path internally disjoint from $\cup_{i=1}^r C_i \cup F$ joining a segment $]c_l^j, u_l^j]_{C_l}$ to a segment $]c_k^{j'}, u_k^{j'}]_{C_k}$, where $1 \le k \le l \le r_1$, $1 \le j \le m_l$ and $1 \le j' \le m_k$.*

Proof of Lemma 4. Suppose that there is a path internally disjoint from $\cup_{i=1}^r C_i \cup F$ joining a vertex $v \in]c_l^j, u_l^j]_{C_l}$ to a vertex $v' \in]c_k^{j'}, u_k^{j'}]_{C_k}$ and choose v and v' so that the sum of the lengths of $]c_l^j, v[_{C_l}$ and $]c_k^{j'}, v'[_{C_k}$ is minimum. Two cases are to take under consider:

Case $k = l$

(1) If v and v' are adjacent. Setting $C' = c_l^j P_{jj'}^l \overleftarrow{]c_l^{j'}, v]}_{C_l} [v', c_l^j[_{C_l}$ in Lemma 1, we obtain: $\alpha(W \cup]c_l^j, v[_{C_l} \cup]c_l^{j'}, v'[_{C_l}) > \alpha(W)$.$(\star)$

On the other hand, by the choice of v and v', $]c_l^j, v[_{C_l}$ and $]c_l^{j'}, v'[_{C_l}$ are path-independent, as they both have property Θ (by the choice of u_l^j and $u_k^{j'}$) then by Lemma 2, we get $\alpha(W \cup]c_l^j, v[_{C_l} \cup]c_l^{j'}, v'[_{C_l}) = \alpha(W)$, which contradicts $(\star)$.

(2) If v and v' are joined by a path $Q_{jj'}^l$ with internal vertices in W. Then taking $C' = c_l^j P_{jj'}^l \overleftarrow{]c_l^{j'}, v]}_{C_l} Q_{j'j}^l [v', c_l^j[_{C_l}$, in Lemma 1, and setting $W_0 = W - Qjj'^l$, we obtain: $\alpha(W \cup]c_l^j, v[_{C_l} \cup]c_l^{j'}, v'[_{C_l}) \ge \alpha(W_0 \cup]c_l^j, v[_{C_l} \cup]c_l^{j'}, v'[_{C_l}) > \alpha(W)$.$(\star\star)$

On the other side by Lemma 2 and because $]c_l^j, v[_{C_l}$ and $]c_l^{j'}, v'[_{C_l}$ are path-independent, and they both have property Θ, we have $\alpha(W \cup]c_l^j, v[_{C_l} \cup]c_l^{j'}, v'[_{C_l}) = \alpha(W)$, so we get a contradiction with the inequality $(\star\star)$.

Case $k \neq l$

(1) If v is adjacent to v'. Setting $C' = c_l^j P_{jj'} \overleftarrow{]c_k^{j'}, v']}_{C_k} (v', v)[v, c_l^j[_{C_l}$ in Lemma 1, we obtain $\alpha(W \cup]c_l^j, v[_{C_l} \cup]c_k^{j'}, v'[_{C_k}) > \alpha(W)$. Furthermore, using Lemma 2 and the fact that $]c_l^j, v[_{C_l}$ and $]c_l^{j'}, v'[_{C_l}$ are path-independent and have property Θ, we get $\alpha(W \cup]c_l^j, v[_{C_l} \cup]c_k^{j'}, v'[_{C_k}) = \alpha(W)$ and hence a contradiction.

(2) If v and v' are joined by a path $Q_{jj'}$ with internal vertices in W. Then taking $C' = c_l^j P_{jj'} \overleftarrow{]c_k^{j'}, v']}_{C_k} Q_{j'j}[v, c_l^j[_{C_l}$, in Lemma 1, and setting $W_0 = W - Qjj'$, we obtain:

$\alpha(W \cup]c_l^j, v[_{C_l} \cup]c_k^{j'}, v'[_{C_k}) \ge \alpha(W_0 \cup]c_l^j, v[_{C_l} \cup]c_k^{j'}, v'[_{C_k}) > \alpha(W)$.$(\star\star\star)$

On the other hand, $]c_l^j, v[_{C_l}$ and $]c_k^{j'}, v'[_{C_k}$ are path independent (by the choice of v and v') and have property Θ (by the choice of u_l^j and $u_k^{j'}$) so Lemma 2 gives: $\alpha(W) = \alpha(W \cup]c_l^j, v[_{C_l} \cup]c_k^{j'}, v'[_{C_k})$ which contradicts $(\star\star\star)$. □

F has no neighbor on the segments $]c_l^j, c_l^{j+1}[_{C_l}$ for $1 \le l \le r_1$ and $1 \le j \le m_l$. So each segment $]c_l^j, u_l^j]_{C_l}$ ($1 \le l \le r_1$, $1 \le j \le m_l$) is independent from F. Furthermore, as showed in Lemma 3, each segment $]c_l^j, u_l^j]_{C_l}$ ($1 \le l \le r_1, 1 \le j \le m_l$) does not verify property Θ and by Lemma 4, all the segments $]c_l^j, u_l^j]_{C_l}$ ($1 \le l \le r_1, 1 \le j \le m_l$)) are pairwise path-independent.

Now, we look at the cycles C_i for $r_1 + 1 \le i \le r_2$. We know that $\alpha(W \cup F)$ increases by at least 1 if a cycle C_i is added but we shall show that it will increase by more if we add more cycles. There are two cases to consider according to whether $C_i - \{c_i\}$ has or not property Θ. Let C_k be a cycle such that $r_1 + 1 \le k \le r_2$.

Lemma 5. *Let $r_1 + 1 \le k \le r_2$ and let C_k be a cycle such that $C_k - \{c_k\}$ does not have property Θ. Let $u_k \in C_k - \{c_k\}$ be the first vertex such that $]c_k, u_k]_{C_k}$ does not have property Θ then*

(a) *$]c_k, u_k]_{C_k}$ is path-independent from each other segment $]c_l, u_l]_{C_l}$ for $r_1 + 1 \le l \le r_2$ verifying the same hypothesis.*

(b) *$]c_k, u_k]_{C_k}$ is path-independent from each segment $]c_l^j, u_l^j]_{C_l}$ for $1 \le l \le r_1$ and $1 \le j \le m_l$.*

Proof of Lemma 5. The proof is similar to the proof of Lemma 4. □

Now if $C_k - \{c_k\}$ has property Θ then we distinguish two cases,

Case 1: $\alpha(W \cup F \cup \{c_k\}) = \alpha(W \cup F)$. Following the orientation of C_k, let u_k be a vertex of $C_k - \{c_k\}$, the nearest to c_k such that $\alpha(W \cup F \cup [c_k, u_k]_{C_k}) > \alpha(W \cup F)$.

Case 2: $\alpha(W \cup F \cup \{c_k\}) > \alpha(W \cup F)$.

We show that

Lemma 6. *In case 1, $[c_k, u_k]_{C_k}$ ($r_1 + 1 \le k \le r_2$) is path-independent from any segment $[c_l, u_l]_{C_l}$ ($r_1 + 1 \le l \le r_2$) or $]c_l, u_l]_{C_l}$ ($r_1 + 1 \le l \le r_2$) or $]c_l^j, u_l^j]_{C_l}$ ($1 \le l \le r_1, 1 \le j \le m_l$).*

In case 2, $\{c_k\}$ is path-independent from any $\{c_l\}$ ($r_1 + 1 \le l \le r_2$) verifying $\alpha(W \cup F \cup \{c_l\}) > \alpha(W \cup F)$ and from any interval of the form $[c_l, u_l]_{C_l}$ ($r_1 + 1 \le l \le r_2$), $]c_l, u_l]_{C_l}$ ($r_1 + 1 \le l \le r_2$) or $]c_l^j, u_l^j]_{C_l}$ ($1 \le l \le r_1, 1 \le j \le m_l$).

Proof of Lemma 6. The proofs are very similar to the proof of Lemma 4 or of Lemma 5.

In case 1, to show that $[c_k, u]_{C_k}$ is path-independent from any segment $[c_l, u_l]_{C_l}$ or $]c_l, u_l]_{C_l}$ or $]c_l^j, u_l^j]_{C_l}$, we suppose to the contrary that a vertex $v \in [c_k, u]_{C_k}$ is joined to a vertex $v' \in [c_l, u_l]_{C_l}$ or $\in\ ]c_l, u_l]_{C_l}$ or $\in\ ]c_l^j, u_l^j]_{C_l}$. In any case and reasoning the same way as Lemma 4, we get a contradiction.

In case 2, we show that $\{c_k\}$ is path-independent from any $\{c_l\}$ verifying the same hypothesis of case 2 or $[c_l, u_l]_{C_l}$ or $]c_l, u_l]_{C_l}$ or $]c_l^j, u_l^j]_{C_l}$. In this case, notice that by hypothesis $[c_k^+, c_k^-]$ has property Θ and is by Lemma 2 path-independent from any segment $[c_l^+, c_l^-]$ of the same type. And the same proof of Lemma 4 gives the desired result. □

In Lemma 6, for $r_1 + 1 \le k \le r_2$, both $[c_k, u_k]_{C_k}$ and $\{c_k\}$ contain a neighbor of F. For technical reasons, we are not going to consider all the cycles C_k ($r_1 + 1 \le k \le r_2$) on which F has exactly one neighbor but only those cycles on which a fixed vertex $z_0 \in F$ has a neighbor. We choose z_0 such that $d_F(z_0) = \delta_F$. We label these cycles from $r_1 + 1$ to r_3 ($r_3 \le r_2$). We observe that

Observation 1. Let k be an integer such that $r_1 + 1 \le k \le r_3$ and such that $C_k - \{c_k\}$ has property Θ. Then z_0 is the only neighbor of c_k in F.

Proof. Suppose to the contrary that $|\Gamma_F(c_k)| \ge 2$, where $\Gamma_F(c_k)$ is the neighborhood of c_k in F. Let $x \in F, x \ne z_0$ be another neighbor of c_k and let P be the path with internal vertices in F joining x and z_0. Then taking $C' = c_k x P z_0 c_k$ in Lemma 1 and the fact that $\alpha(W \cup C_k - \{c_k\}) = \alpha(W)$ gives a contradiction. ■

Observation 2. If z_0 belongs to every maximum independent set S of F, then for $k, r_1 + 1 \le k \le r_3$ there does not exist segments I_i of type $[c_k, u_k]_{C_k}$ or $\{c_k\}$ (as defined above) such that $\alpha(W \cup F \cup I_i) \ge \alpha(W \cup F) + 1$.

Proof. Suppose that z_0 is contained in every maximum independent set S of F and that there exists a segment I_i of type $[c_k, u_k]_{C_k}$ or $\{c_k\}$ such that $\alpha(W \cup F \cup I_i) \ge \alpha(W \cup F) + 1$. Clearly by the minimality of I_i, c_k is contained in a maximum independent set of $W \cup F \cup I_i$.

(1) If $I_i = [c_k, u_k]_{C_k}$ for some $k, r_1 + 1 \le k \le r_3$.

Let $S_{\max}(W \cup F \cup I_i)$ be a maximum independent set of $W \cup F \cup I_i$. $S_{\max}(W \cup F \cup I_i)$ contains either c_k or z_0 but not both. So $|S_{\max}(W \cup F \cup I_i)| = |S_{\max}(W \cup F - \{z_0\} \cup I_i)| = |S_{\max}(W \cup F \cup I_i - \{c_k\})|$ hence $\alpha(W \cup F \cup I_i - \{c_k\}) = \alpha(W \cup F \cup I_i) > \alpha(W \cup F)$ and this is a contradiction because $I_i - \{c_k\} \subset C_k - \{c_k\}$ and by hypothesis an interval $I_i = [c_k, u_k]_{C_k}$ is chosen when $C_k - \{c_k\}$ verifies property Θ.

(2) If $I_i = \{c_k\}$ for some $k, r_1 + 1 \le k \le r_3$. Here again either c_k or z_0, but not both, belong to a maximum independent set of $W \cup F \cup I_i$, so $\alpha(W \cup F) = \alpha(W \cup F \cup I_i)$ and this gives a contradiction with $\alpha(W \cup F \cup I_i) > \alpha(W \cup F)$. ■

To summarize, we have showed that on every cycle C_k, for $1 \leq k \leq r_3$, there is a segment I_i or m_k segments I_i, which if added will increase the independence number of $W \cup F$. We have showed that these segments are pairwise path-independent. To achieve the proof of Theorem 2, we look at two cases:

(1) If every maximum independent set S of F contains z_0, then by Observation 2, we have only segments I_i of type $]c_k, u_k]_{C_k}$ (for $r_1 + 1 \leq k \leq r_3$) or $]c_k^j, u_k^j]_{C_k}$ (for $1 \leq k \leq r_1$, $1 \leq j \leq m_k$) that verify $\alpha(W \cup F \cup I_i) > \alpha(W \cup F)$. Notice that these segments are independent from F and pairwise path-independent (by Lemmas 4 and 5). Set $m = \sum_{k=1}^{r_1} m_k$ and let H^i be the union of the components of W that contain a neighbor of I_i. We have:

$\alpha(W \cup F \cup \bigcup_{k=1}^{r_3} C_k) \geq \alpha(W \cup F \cup \bigcup_{i=1}^{m+r_3-r_1} I_i) = \alpha(W - \bigcup_{i=1}^{m+r_3-r_1} H^i) + \sum_{i=1}^{m+r_3-r_1} \alpha(H^i \cup I_i) + \alpha(F) \geq \alpha(W - \bigcup_{i=1}^{m+r_3-r_1} H^i) + \sum_{i=1}^{m+r_3-r_1} \alpha(H^i) + \alpha(F) + m + r_3 - r_1 = \alpha(W \cup F) + m + r_3 - r_1$.

(2) If there exists a maximum independent set S of F such that $z_0 \notin S$, then put $F' = F - \{z_0\}$. We have $\alpha(F) = \alpha(F')$. Notice that using Observation 1, any segment I_i is independent from F'. Furthermore, by Lemmas 4–6, all the segments I_i are pairwise path-independent. Replacing F by F' in the proof of the previous case, we get

$\alpha \geq \alpha(W \cup F' \cup \bigcup_{i=1}^{r_3} C_i) \geq \alpha(W \cup F') + r_3 - r_1 + m = \alpha(W \cup F) + r_3 - r_1 + m.$

Finally, by the choice of z_0, we have that $r_3 - r_1 + m \geq d(z_0) - 1 \geq \delta - 1$ and it follows that in both case (1) and case (2) we have

$\alpha \geq \alpha(W \cup F \cup \bigcup_{i=1}^{r_2} C_i) \geq \alpha(W \cup F) + \delta - 1$

So $\alpha(G - \bigcup_{i=1}^{r} C_i) \leq \alpha - \delta + 1$ and this completes the proof of Theorem 2.

Proof of Theorem 1. According to Proposition 1, $G - \bigcup_{i=1}^{r} C_i$ can be covered by at most $\alpha(G - \bigcup_{i=1}^{r} C_i)$ vertex-disjoint components that are cycles, edges or vertices. Denote by $\mathcal{E}$ the set of these components. By Theorem 2, the number of these components is at most $\alpha - \delta + 1$. Finally, $\mathcal{F} \cup \mathcal{E}$ is a pseudo 2-factor of G with at most $\alpha - \delta + 1$ components that are edges or vertices and this completes the proof of Theorem 1. ■

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References

[1] T. Andreae, On independent cycles and edges in graphs, Discrete Math. 149 (1996) 291–297.
[2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan & Co., London, 1976.
[3] J.A. Bondy, Basic graph theory: Paths and circuits, in: R.L. Graham, M. Grötschel, L. Lovasz (Eds.), Handbook of Combinatorics, North-Holland, Amsterdam, 1993.
[4] Z. Danhong, W. Hong, A minimum degree result for disjoint cycles and forests in bipartite graphs, Australas. J. Combin. 29 (2004) 35–47.
[5] S. Fujita, Recent results on disjoint cycles in graphs, Electron. Notes Discrete Math. 22 (2005) 409–412.
[6] M. Kouider, Cycles in graphs with prescribed stability number and connectivity, J. Combin. Theory Ser. B 60 (1994) 315–318.
[7] L. Lesniak, Independent cycles in graphs, J. Combin. Math. Combin. Comput. 17 (1995) 55–63.
[8] T. Niessen, Minimum degree, independence number and regular factors, Graphs Combin. 11 (1995) 376–378.
[9] G. Schuster, A minimum degree result for disjoint cycles and forests in graphs, Combinatorica 18 (1998) 425–436.